On the notion of Necessary and Possibly Interactions in MCDA

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Example (Classic example)

	1 : Mathematics (M)	2 : Statistics (S)	3 : Language (L)
a	16	13	7
b	16	11	9
c	6	13	7
d	6	11	9

- for a student "good" in Mathematics, Language is more important than Statistics

$$\Longrightarrow b \; P \; a, \tag{1}$$

- for a student "bad" in Mathematics, Statistics is more important than Language

$$\Longrightarrow c \; P \; d. \tag{2}$$

These preferences are not representable by an additive model.

We need a non-additive model

- Which model we have to choose?
- Here, we choose an extension of an additive model allowing interaction among criteria: a 2-additive Choquet integral.

Definition

For any $z := (z_1, \ldots, z_n) \in \mathbb{R}^+$, the expression of the 2-additive Choquet integral is:

$$C_\mu(z_1, \ldots, z_n) = \sum_{i=1}^{n} V_i^\mu z_i - \frac{1}{2} \sum_{\{i,j\} \subseteq N} I_{ij}^\mu \, |z_i - z_j| \tag{3}$$

where

- $V_i^\mu \equiv$ Shapley value of i, is the importance of the criterion i
- $I_{ij}^\mu = \mu(\{i,j\}) - \mu(\{i\}) - \mu(\{j\})$ is the interaction index between criteria i and j (w.r.t. μ).

A 2-additive Choquet integral

- It seems a good compromise between the arithmetic mean and the general Choquet integral;
- It assumes that only interactions between two criteria are meaningful;
- Its interaction index I_{ij}^{μ} is not difficult to interpret (**Really?**):
  - $I_{ij}^{\mu} = \mu(\{i,j\}) - \mu(\{i\}) - \mu(\{j\} > 0 \Longrightarrow$ *complementary between i and j*
  - $I_{ij}^{\mu} = \mu(\{i,j\}) - \mu(\{i\}) - \mu(\{j\} < 0 \Longrightarrow$ *substitutability between i and j*
  - $I_{ij}^{\mu} = \mu(\{i,j\}) - \mu(\{i\}) - \mu(\{j\} = 0 \Longrightarrow$ *Independence between i and j*

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	Par.1	Par.2	Par.3	Par.4	Par.5	Par.6	Par.7	Par.8
$C_\mu(a)$	8.5	13.75	9.1	13.765	13.75	13.75	11.47	12.535
$C_\mu(b)$	9.5	14.25	9.7	13.995	14.25	14.25	11.93	12.785
$C_\mu(c)$	7.75	9.75	7.75	11.325	11.25	9.75	9.45	9.515
$C_\mu(d)$	7.25	9.25	7.25	10.295	9.75	9.25	8.91	9.265
μ_M	0	0.75	0	0.685	0.75	0.75	0.36	0.485
μ_S	0.25	0.5	0.25	0.73	0.75	0.5	0.465	0.455
μ_L	0	0.25	0	0.315	0	0	0.205	0.32
μ_{MS}	0.25	0.75	0.35	0.785	0.75	0.75	0.565	0.68
μ_{ML}	0.75	1	0.65	1	0.1	0.75	0.805	0.795
μ_{SL}	0.25	0.75	0.25	0.945	0.75	0.75	0.66	0.785
V_M^μ	0.375	0.5	0.375	0.37	0.5	0.5	0.35	0.35
V_S^μ	0.25	0.25	0.3	0.365	0.375	0.375	0.33	0.33
V_L^μ	0.375	0.25	0.325	0.265	0.125	0.125	0.32	0.32
I_{MS}^μ	0	-0.5	0.1	-0.63	-0.75	-0.5	-0.26	-0.26
I_{ML}^μ	0.75	0	0.65	0	0.25	0	0.24	-0.01
I_{SL}^μ	0	0	0	-0.1	0	0.25	-0.01	0.01

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	Par.9	Par.10	Par.11	Par.12	Par.13	Par.14	Par.15
$C_\mu(a)$	15.55	10.627	10.45	9.28	11.41	5.29	12.655
$C_\mu(b)$	15.65	10.749	10.75	9.76	11.91	7.35	12.825
$C_\mu(c)$	10.3	8.814	7.85	7.96	9.39	6.91	9.635
$C_\mu(d)$	10.1	8.01	7.55	7.4	8.89	6.81	9.305
μ_M	0.95	0.135	0.15	0	0.36	0	0.485
μ_S	0.55	0.402	0.25	0.28	0.455	0.195	0.475
μ_L	0.45	0.07	0	0.01	0.195	0.115	0.32
μ_{MS}	0.95	0.537	0.5	0.38	0.555	0.195	0.7
μ_{ML}	1	0.668	0.55	0.63	0.795	0.655	0.795
μ_{SL}	1	0.402	0.35	0.28	0.66	0.46	0.785
V_M^μ	0.475	0.3665	0.4	0.36	0.35	0.27	0.35
V_S^μ	0.275	0.367	0.35	0.325	0.33	0.27	0.34
V_L^μ	0.25	0.2665	0.25	0.315	0.32	0.46	0.31
I_{MS}^μ	-0.55	0	0.1	0.1	-0.26	0	-0.26
I_{ML}^μ	-0.4	0.463	0.4	0.62	0.24	0.54	-0.01
I_{SL}^μ	0	-0.07	0.1	-0.01	0.01	0.15	-0.01

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- In this example, it seems clear that it is not easy to interpret the interaction between two criteria.
- Could we conclude that the subjects Mathematics and Statistics are complementary, redundant or independent? **Answering this question is not obvious**.
- In fact, the only information provided by the preferences $b\ P\ a$ and $c\ P\ d$ is that: "the three criteria (subjects) are not independent" i.e. **the three interaction indices cannot be null simultaneously**.

Definition (Necessary and Possibly interaction)

Let be $i, j \in N$, $i \neq j$ and $\mathcal{C}_{pref}$ the set of all capacities compatible with a preference information given by the DM.

1. There exists a **possible positive (respectively negative) interaction** between i and j if there exists a capacity $\mu \in \mathcal{C}_{pref}$ such that $I_{ij}^{\mu} > 0$ (respectively $I_{ij}^{\mu} < 0$).
2. There exists a **necessary positive (respectively negative) interaction** between i and j if $I_{ij}^{\mu} > 0$ (respectively $I_{ij}^{\mu} < 0$) for all capacity $\mu \in \mathcal{C}_{pref}$.
3. i and j are **possibly without interaction** if there exists a capacity $\mu \in \mathcal{C}_{pref}$ such that $I_{ij}^{\mu} = 0$.
4. i and j are **necessary without interaction** if $I_{ij}^{\mu} = 0$ for all capacity $\mu \in \mathcal{C}_{pref}$.

Hypotheses

- DM is able to identify two reference levels: $\begin{cases} \mathbf{0}_i \in X_i \text{ that is neutral} \\ \mathbf{1}_i \in X_i \text{ that is satisfactory} \end{cases}$ on each attribute i
- The DM is able to give a preference information $\{P, I\}$ on the following set of binary actions (alternatives) $\mathcal{B}$

Definition

A *binary action* is an element of the set

$$\mathcal{B} = \{\mathbf{0}_N,\ (\mathbf{1}_i, \mathbf{0}_{N-i}),\ (\mathbf{1}_{ij}, \mathbf{0}_{N-ij}),\ i, j \in N, i \neq j\}$$

where

- $\mathbf{0}_N = (\mathbf{1}_\emptyset, \mathbf{0}_N) =: a_0$ is the action considered neutral on all criteria.
- $(\mathbf{1}_i, \mathbf{0}_{N-i}) =: a_i$ is an action considered satisfactory on criterion i and neutral on the other criteria.
- $(\mathbf{1}_{ij}, \mathbf{0}_{N-ij}) =: a_{ij}$ is an action considered satisfactory on criteria i and j and neutral on the other criteria.

Theorem (The case $I = \emptyset$)

There always exists a possible positive interaction between two criteria i and j, i.e.

- *i and j are not necessary without interaction.*
- *i and j are not necessary interact negatively.*

Theorem (The case $I \neq \emptyset$)

The interaction between two criteria i and j is necessary negative

$$\Updownarrow$$

$[a_{ij} \sim a_i$ and $a_j \ TC_P \ a_0]$ (2-MOPI property).

Example (A particular case)

$[a_{ij} \ I \ a_i$ and $a_j \ P \ a_0] \Longrightarrow$ the interaction between i and j is necessary negative.